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A VARIATIONAL FORMULATION FOR PLATE BUCKLING PROBLEMS BY THE HYBRID FINITE ELEMENT METHOD

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Abstract—A functional is derived for development of stress hybrid finite elements for plate buckling problems. The equilibrium equations inside the element are identically satisfied in terms of Southwell stress functions *and* the transverse displacement. Along the boundary of the element further displacement and normal slope functions are employed. These functions are so chosen as to satisfy the interelement compatibility requirements when the elements are connected. The boundary and internal displacements are selected entirely independently and comments are made on the choice of interpolation functions for the internal displacement.

The stationarity of the functional is shown to lead to satisfaction of the equilibrium conditions along interelement boundaries, and the compatibility conditions inside the elements. The paper includes the details of a simple rectangular element and the results of a number of plate buckling problems analysed by the developed element.

INTRODUCTION

The hybrid stress finite element model, as proposed by Pian in 1964[1], has proved to be a very versatile element for a variety of applications. The salient features of this element include the flexibility with which the elemental equations can be formed, the direct determination of stresses and a facility for satisfying the force as well as the kinematic boundary conditions. A detailed review of the hybrid element and its applications has been given by Pian[2]. The essential features of the formulation consist of the explicit satisfaction of equilibrium equations within the elements and the compatibility requirements on inter-element surfaces as well as on surfaces where kinematic boundary conditions are prescribed. To this end one generally employs equilibrating stress fields within an element and an independent set of displacement fields on the element's surface. The remaining requirements of the true solution, namely the compatibility conditions within the elements and the satisfaction of equilibrium equations on inter-element surfaces and on surfaces where force boundary conditions are prescribed, tend to be satisfied implicitly through the process of extremisation.

In its original form the variational formulation of Pian was limited to static equilibrium problems. Subsequently the pertinent functional was generalised by Tabarrok to include the dynamic effects[3, 4]. For stability analysis too, some investigators have attempted to develop hybrid elements, e.g. Allman[5, 6] and Tong *et al.*[7]. However strictly these latter formulations are based on variants of a mixed functional and they fail to preserve the essential features of the true hybrid finite element model. In this paper we outline a novel procedure for developing stress hybrid models for elastic stability problems of plates. to illustrate the procedure we develop a simple rectangular element by means of which we analyse several buckling problems.

FORMULATION

Consider a flat plate subjected to inplane loads N_{nn} and N_{nt} along its boundary (see Fig. 1). It is assumed that these prescribed boundary forces result in known inplane forces N_{ij} within the plate and that $N_{ij,j} = 0$.

Now it can be shown that the compatibility equations for the plate are obtained as the stationary conditions of the complementary energy principle[8]

$$\pi_c = V - U^* + \int_{s_u} (\bar{w} \cdot \mathscr{V}_n - \bar{w}_{,n} M_{nn})\, ds - \sum_{s_u} \bar{w} \int_{s-}^{s+} \mathscr{V}_n\, ds \tag{1}$$

where U^* is the complementary strain energy and it is a function of bending moments M_{ij}. V is the potential energy of inplane forces and it is a function of displacement gradients $w_{,i}$. $\mathscr{V}_n$ is the effective shear force (including the contribution of inplane forces). M_{nn} is the normal bending moment, and s_u denotes that part of the boundary on which kinematic quantities are prescribed. The summation in eqn (1) refers to the work done at the corners of the boundary s_u. The functional π_c involves both the moments M_{ij} and the displacement gradients $w_{,i}$. However these quantities are not independent and their variations must satisfy the equilibrium equation

$$M_{ij,ij} = (N_{ij} w_{,i})_{,j}. \tag{2}$$

Furthermore M_{nn} and $\mathscr{V}_n$ are required to satisfy the force boundary conditions along s_τ $(= s - s_u)$, i.e.

$$\left.\begin{aligned} M_{nn} &= \bar{M}_{nn} \\ \mathscr{V}_n &= \bar{\mathscr{V}}_n. \end{aligned}\right\} \quad \text{along } s_T \tag{3}$$

We may now modify π_c, in the manner described by Tong and Pian[9], for purposes of developing hybrid elements. To this end we first consider the domain to be made up of p elements within each of which eqn (2) is satisfied identically. However, along inter-element boundaries, which we denote by s_n, and along s_τ we relax the explicit satisfaction of equilibrium conditions. Instead we pose these requirements as a set of constraints. Thus over the common boundary of elements I and II we stipulate that

$$\left.\begin{aligned} (M_{nn})_I + (M_{nn})_{II} &= 0 \\ (\mathscr{V}_n)_I + (\mathscr{V}_n)_{II} &= 0. \end{aligned}\right\} \quad \text{on } s_{np\,I,II} \tag{4}$$

The constraint eqns (3) and (4) may now be appended to π_c by Lagrange multipliers $w_{b,n}$ and w_b. Thus the modified functional, for p elements, may now be written as

$$\pi_c' = \sum_p \Big[V_p - U_p^* + \int_{s_{up}} (\bar{w} \cdot \mathscr{V}_n - \bar{w}_{,n} M_{nn})\, ds - \sum_{s_{up}} \bar{w} \int_{s-}^{s+} \mathscr{V}_n\, ds + \int_{s_{np}} (w_b \mathscr{V}_n - w_{b,n} M_{nn})\, ds$$
$$+ \sum_{s_{np}} w_b \int_{s-}^{s+} \mathscr{V}_n\, ds + \int_{s_{\tau p}} (w_b[\mathscr{V}_n - \bar{\mathscr{V}}_n] + w_{b,n}[M_{nn} - \bar{M}_{nn}])\, ds + \sum_{s_{\tau p}} w_b \int_{s-}^{s+} (\mathscr{V}_n - \bar{\mathscr{V}}_n)\, ds \Big]. \tag{5}$$

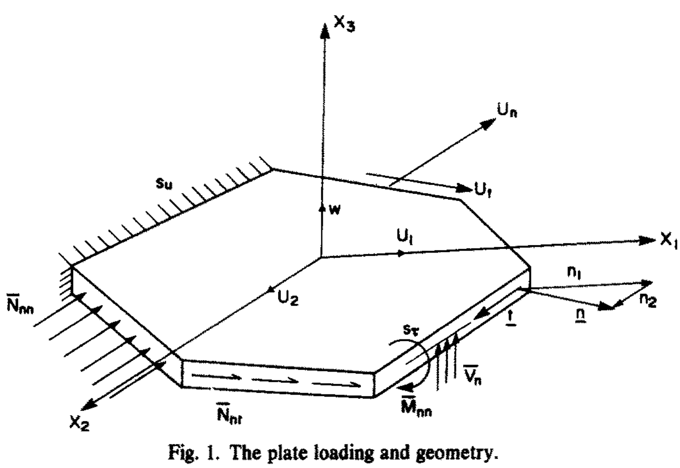

Fig. 1. The plate loading and geometry.

It is not difficult to see from eqn (5) that the Lagrange multipliers w_b and $w_{b,n}$ turn out to be the boundary displacement and normal slope respectively, when π_c' is extremised. Accepting this interpretation *à priori* and noting that

$$s_p = s_{up} + s_{np} + s_{\tau p} \tag{6}$$

we may rewrite our functional as

$$\pi_H = \sum_p \left[V_p - U_p^* + \int_{s_p} (w_b \mathcal{V}_n - w_{b,n} M_{nn})\, ds - \int_{s_{\tau p}} (w_b \bar{\mathcal{V}}_n - w_{b,n} \bar{M}_{nn})\, ds \right.$$
$$\left. + \sum_{s_p} w_b \int_{s-}^{s+} \mathcal{V}_n\, ds - \sum_{s_{\tau p}} w_b \int_{s-}^{s+} \bar{\mathcal{V}}_n\, ds \right]. \tag{7}$$

In π_H we have the required functional for developing hybrid elements. The quantities that appear in π_H are; M_{ij}, w, w_b, $w_{b,n}$. It is to be noted that the displacement inside the element, i.e. w, is independent of the boundary displacement w_b and normal slope $w_{b,n}$. The admissibility requirements of the pertinent variables, apart from the continuity requirement for integrability of π_H, are (i) M_{ij} and w must satisfy the equilibrium eqn (2) (ii) w_b, $w_{b,n}$ must satisfy compatibility requirements along inter-element boundaries and on s_u.

Theorem: amongst all admissible fields of bending moments (M_{ij}), displacements (w) and boundary displacements and slopes (w_b, $w_{b,n}$) the true solution is distinguished by the stationary condition of π_H.

To prove the theorem we equate the first variation of π_H to zero and thence deduce the conditions associated with the true solution. Thus

$$\delta\pi_H = \sum_p \left[-\delta U_p^* + \delta V_p - \int_{s_p} (w_{b,n}\delta M_{nn} - w_b \delta \mathcal{V}_n)\, ds \right.$$
$$- \int_{s_p} (M_{nn}\delta w_{b,n} - \mathcal{V}_n \delta w_b)\, ds + \int_{s_{\tau p}} (\bar{M}_{nn}\delta w_{b,n} - \bar{\mathcal{V}}_n \delta w_b)\, ds$$
$$\left. + \sum_{s_p} \left(w_b \delta \int_{s-}^{s+} \mathcal{V}_n\, ds + \delta w_b \int_{s-}^{s+} \mathcal{V}_n\, ds \right) - \sum_{s_{\tau p}} \delta w_b \int_{s-}^{s+} \bar{\mathcal{V}}_n\, ds \right] = 0. \tag{8}$$

Now on invoking eqn (6) and recognising that on s_u the variations of w_b and $w_{b,n}$ vanish, we may rewrite eqn (8) as follows

$$\delta\pi_H = \sum_p \left[-\delta U_p^* + \delta V_p - \int_{s_p} (w_{b,n}\delta M_{nn} - w_b \delta \mathcal{V}_n)\, ds \right.$$
$$+ \int_{s_{\tau p}} [(\bar{M}_{nn} - M_{nn})\delta w_{b,n} - (\bar{\mathcal{V}}_n - \mathcal{V}_n)\delta w_b]\, ds$$
$$- \int_{s_{np}} (M_{nn}\delta w_{b,n} - \mathcal{V}_n \delta w_b)\, ds + \sum_{s_{\tau p}} \delta w_b \int_{s-}^{s+} (\mathcal{V}_n - \bar{\mathcal{V}}_n)\, ds$$
$$\left. + \sum_{s_{np}} \delta w_b \int_{s-}^{s+} \mathcal{V}_n\, ds + \sum_{s_p} w_b \delta \int_{s-}^{s+} \mathcal{V}_n\, ds \right] = 0. \tag{9}$$

It is now apparent that the vanishing coefficients for $\delta w_{b,n}$ and δw_b on $s_{\tau p}$ yield the force boundary conditions on $s_{\tau p}$. On s_{np} the variations of w_b and $w_{b,n}$ are not independent in view of the inter-element compatibility requirements that they must satisfy. Hence in this case we require that

$$\sum_p \left[\int_{s_{np}} (M_{nn}\delta w_{b,n} - \mathcal{V}_n \delta w_b)\, ds + \sum \delta w_b \int_{s-}^{s+} \mathcal{V}_n\, ds \right] = 0. \tag{10}$$

Above equation implies that the virtual work of boundary forces on all elements must vanish.

Clearly this is the condition of inter-element equilibrium. It now remains to consider the terms

$$\sum_p \left[-\delta U_p^* + \delta V_p - \int_{s_p} (w_{b,n}\delta M_{nn} - w_b\delta \mathcal{V}_n)\,ds + \sum_{s_p} w_b\delta \int_{s-}^{s+} \mathcal{V}_n\,ds\right] = 0 \tag{11}$$

where

$$\delta U_p^* = \int_{A_p} k_{ij}\delta M_{ij}\,dx_1\,dx_2, \qquad k_{ij} = \text{curvature tensor}$$

and

$$\delta V = \int_{A_p} \bar{N}_{ij}w_{,j}\delta w_{,i}\,dx_1\,dx_2, \quad A_p = \text{area of } p\text{th element.} \tag{12}$$

But variations of M_{ij} and $w_{,i}$ are not independent and they must satisfy the equilibrium eqn (2). To facilitate the satisfaction of this constraint we will use a modified form of Southwell's stress functions U_1, U_2[12]. We let

$$M_{11} = U_{2,2} + N_{11}w \qquad M_{22} = U_{1,1} + N_{22}w$$
$$2M_{12} = -(U_{12} + U_{2,1}) + 2N_{12}w. \tag{13}$$

Along a straight boundary one can show that normal bending moment and effective shear may then be expressed as [8]

$$M_{nn} = U_{t,t} + N_{nn}w \qquad \mathcal{V}_n = -U_{n,tt} + (N_{nt}w)_{,t} \tag{14}$$

here U_n and U_t are the stress functions resolved along the normal and tangent of the boundary. Thus in terms of the direction cosines of the normal, n_i, and those of the tangent, t_i, we have

$$U_n = n_iU_i \qquad U_t = t_iU_i. \tag{15}$$

Now substituting from eqns (12)–(14) into eqn (11), we obtain

$$\begin{aligned}\sum_p \Big[&-\int_{A_p} k_{11}\delta(U_{2,2} + N_{11}w) + k_{22}\delta(U_{1,1} + N_{22}w) - k_{12}\delta(U_{1,2} + U_{2,1} - 2N_{12}w)\,dx_1\,dx_2 \\ &+ \int_{A_p} [N_{11}w_{,1}\delta w_{,1} + N_{22}w_{,2}\delta w_{,2} + N_{12}(w_{,1}\delta w_{,2} + w_{,2}\delta w_{,1})]\,dx_1\,dx_2 \\ &- \int_{s_p} [w_{b,n}\delta(U_{t,t} + N_{nn}w) + w_b\delta(U_{n,tt} - (N_{nt}w)_{,t})]\,ds \\ &+ \sum_p w_b\delta \int_{s^-}^{s^+} (-U_{n,t} + N_{nt}w)_{,t}\,ds\Big] = 0.\end{aligned} \tag{16}$$

On carrying out the integrations by parts and noting that the last term in eqn (16) is an exact differential we obtain the extremum conditions as follows

In A_p

$$\begin{aligned}&(k_{22,1} - k_{12,2})\delta U_1 = 0 \\ &(k_{11,2} - k_{12,1})\delta U_2 = 0 \\ &[N_{11}(w_{,11} - k_{11}) + N_{22}(w_{,22} - k_{22}) + 2N_{12}(w_{,12} - k_{12})]\delta w = 0.\end{aligned} \tag{17}$$

Along S_p

$$(-k_{tt} - w_{b,tt})\delta U_n = 0$$
$$(w_{b,nt} + k_{nt})\delta U_t = 0 \tag{18}$$
$$[N_{nn}(w_{,n} - w_{b,n}) + N_{nt}(w_{,t} - w_{b,t})]\delta w = 0$$
$$\sum_{s_p} [w_{b,n}\delta U_t - w_{b,t}\delta U_n]_{s-}^{s+} = 0.$$

The physical meaning of eqn (17) is self-evident. It is of interest also to note that eqns (18) imply that for the true solution, some of the curvatures and slopes of the interior moments and displacement fields must match those of the independently assumed boundary displacements. However it is not required for the absolute values of the interior and boundary displacements to become identical at any point on the boundary, i.e. the interior displacement field w, evaluated along a boundary, may differ from that of the boundary displacement w_b, by a constant.

The meaning of the corner terms becomes evident if we transform from the normal-tangent axes back to x_1x_2 axes. Then the last equation of (18) may be written as

$$\sum_{p} (n_1{}^2 + n_2{}^2)[w_{b,x_1}\delta U_2 - w_{b,x_2}\delta U_1]_{s-}^{s+} = 0. \tag{19}$$

From this equation it can be seen that the corner term is independent of the orientations of the two edges meeting at the corner. Further, recognising the continuity and arbitrariness of U_1 and U_2, at the corner, we note that eqn (19) implies that for the exact solution, w_{b,x_1} and w_{b,x_2} will be continuous at the corner.

FINITE ELEMENT FORMULATION

As in the standard hybrid element we require equilibrating stress fields within the element. In the present formulation we also expand for the internal displacement w. Indeed we may *view* the w field as an "extra stress field" involved in the equilibrium equation. Hence we expand the vector of moments and the displacement in terms of some polynomial functions with unspecified coefficients $\{\beta\}$, viz

$$\begin{bmatrix} M \\ w \end{bmatrix} = \begin{bmatrix} A_{MM} & \lambda A_{ww} \\ 0 & A_{ww} \end{bmatrix} \begin{bmatrix} \beta_M \\ \beta_w \end{bmatrix} \tag{20}$$

where $\{M\}^t = [M_{11}M_{22}M_{12}]$. The inplane forces appear explicitly in the partitioned matrix A_{ww}. It is convenient to specify a relative distribution for the inplane forces and denote the magnitude of the given distribution by λ. Then the scalar λ can be factored out of A_{ww} as shown in eqn (20). Next we interpolate for all the boundary displacements in terms of some nodal displacement vector $\{q\}$, i.e. we write

$$\begin{bmatrix} w_b \\ w_{b,n} \end{bmatrix} = [L]\{q\} \tag{21}$$

clearly $[L]$ contains the interpolation functions in terms of x_1 and x_2. It is worth pointing out again that the expansion for the interior displacement in eqn (20) is completely independent of that for the boundary displacements in eqn (21).

Finally we substitute from eqns (20) and (21) into eqn (7) to obtain the following discretised form of π_H:

$$\pi_H = \sum_p \frac{1}{2}[\beta_M\beta_w]\left(\begin{bmatrix} 0 & 0 \\ 0 & \lambda C_{ww} \end{bmatrix}\begin{bmatrix} \beta_M \\ \beta_w \end{bmatrix} - \frac{1}{D(1-\nu^2)}\begin{bmatrix} B_{MM} & \lambda B_{Mw} \\ \lambda B_{Mw}^t & \lambda^2 B_{ww} \end{bmatrix}\begin{bmatrix} \beta_M \\ \beta_w \end{bmatrix} - \begin{bmatrix} R_M \\ \lambda R_w \end{bmatrix}\{q\}\right) + \{\bar{F}\}^t\{q\} \tag{22}$$

where the matrices $[B]$, $[C]$ and $[R]$ arise in the evaluations of U^*, V and the work of the interior stresses along the entire boundary of the element, respectively.† The matrix $\{\bar{F}\}$ is obtained in the evaluation of the work of the prescribed forces on $s_{\tau p}$ and it can therefore be regarded as the vector of the generalised forces. Now the variations of β in π_H give rise to the following Euler-Lagrange equations, for each element.

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & \lambda C_{ww} \end{bmatrix} - \frac{1}{D(1-\nu^2)}\begin{bmatrix} B_{MM} & \lambda B_{Mw} \\ \lambda B_{Mw}^t & \lambda^2 B_{ww} \end{bmatrix}\right)\begin{bmatrix} \beta_M \\ \beta_w \end{bmatrix} = \begin{bmatrix} R_M \\ \lambda R_w \end{bmatrix}\{q\}. \tag{23}$$

On denoting the symmetric coefficient matrix of the β's by $[E]$ we may express eqn (23), symbolically as follows

$$\{\beta\} = [E]^{-1}[R]\{q\}. \tag{24}$$

Substituting now from eqn (24) into eqn (22) we may express π_H as follows

$$\pi_H = \sum_p -\frac{1}{2}\{q\}^t[K]\{q\} + \{\bar{F}\}^t\{q\} \tag{25}$$

where

$$[K] = [R][E]^{-1}[R]. \tag{26}$$

Before evaluating the variations of π_H with respect to the q's we will transform from the elemental displacement nodes to a set of independent global nodes q^*'s. This process which is referred to as the connection or assembly of elements, is well known[10]. Subsequent to the process of connection, π_H may be written as

$$\pi_H = -\frac{1}{2}\{q\}^{*t}[K]^*\{q\}^* + \{F\}^{*t}\{q\}^* \tag{27}$$

where all the starred quantities are associated with the global system. Finally on equating the variation of π_H to zero we obtain the system equations as

$$[K(\lambda)]^*\{q\}^* = \{\bar{F}\}^*. \tag{28}$$

Equation (28) expresses the force displacement relation for the system and it is now evident that $[K(\lambda)]^*$ is the system stiffness matrix. It should be noted that $[K(\lambda)]^*$ is a function of λ, i.e. the elements of this matrix depend upon the magnitude and distributions of the inplane forces. Plate's loss of stability is characterised by singularities of $[K(\lambda)]^*$ and by plotting the determinant of $[K(\lambda)]^*$ against λ one may locate the zeros of the determinant and obtain the critical loads.

THE CHOICE OF INTERPOLATION FUNCTIONS

The selection of interpolation functions for the boundary displacements and slopes is straight forward and a number of such functions have been examined by Pian[11]. However the selection of interpolation functions for the interior displacement and the moments requires some care. These functions are best obtained via eqns (13) by initially writing interpolation functions for the stress functions U_1 and U_2 as well as the displacement w and then deriving those of the moments.

Now we pose the following question. Amongst the possible interpolation functions for the moments and the interior displacement are there some that will render $|E| \equiv 0$, independent of the value of λ? The answer is in the affirmative and as a result it is necessary to recognise and suppress these particular functions, otherwise the stiffness matrix of the element cannot be formed. From the foregoing it can be deduced that matrix $[E]$ will become singular for any

†(See illustrative example).

value of λ, if the complementary strain energy U^* and the potential energy of inplane forces, V, vanish simultaneously. Thus for $[E]$ to remain non singular at least one of these energy functions must remain positive definite. Now the complementary strain energy, when expressed in terms of moments, is a positive definite function i.e. it vanishes only when $M_{11} = M_{22} = M_{12} = 0$. However when expressed in terms of U_1, U_2 and w, this energy function is no longer positive definite, i.e. it may vanish for some non trivial forms of U_1, U_2 and w. To determine these particular forms we equate the moments to zero and from eqns (13), determine the specific forms of U_1, U_2 and w. Thus for M_{11} and M_{22} to vanish we have

$$U_2 = -\int \bar{N}_{11} w \, \mathrm{d}x_2 + f(x_1)$$

$$U_1 = -\int \bar{N}_{22} w \, \mathrm{d}x_1 + g(x_2)$$

where $f(x_1)$ and $g(x_2)$ are pure functions of x_1 and x_2 respectively. Substituting into the expression for M_{12} and requiring this moment to vanish we obtain

$$2\bar{N}_{12} w + \frac{\mathrm{d}}{\mathrm{d}x_1} \int \bar{N}_{11} w \, \mathrm{d}x_2 + \frac{\mathrm{d}}{\mathrm{d}x_2} \int \bar{N}_{22} w \, \mathrm{d}x_1 = \frac{\mathrm{d}g}{\mathrm{d}x_2} + \frac{\mathrm{d}f}{\mathrm{d}x_1} \tag{29}$$

we can cast this equation in a more convenient form by differentiating it once with respect to x_1 and once with respect to x_2. We then obtain

$$2\bar{N}_{12} w_{,12} + \bar{N}_{11} w_{,11} + \bar{N}_{22} w_{,22} = 0. \tag{30}$$

Equation (30) is identical to the right hand side of eqn (2)—bearing in mind the equilibrium of inplane forces. thus such forms of w which do not contribute to the equilibrium equation will also not contribute to the complementary strain energy. For the special case of $w \equiv 0$ eqn (30) will clearly be satisfied and hence the complementary strain energy function may vanish for non trivial forms of U_1 and U_2. From eqn (29) it is readily seen that such possible forms of U_1 and U_2 are given by

$$U_1 = c_1 + c_2 x_2 \qquad U_2 = c_3 - c_2 x_1. \tag{31}$$

These expressions are analogous to the rigid body modes of inplane displacements of plates [12] and they can be readily suppressed by excluding them from the interpolation functions of U_1 and U_2. Once these particular U_1 and U_2 modes are suppressed the positive definiteness of the complementary strain energy function will depend upon some non trivial forms of w only. When all three inplane forces are present it can be seen that eqn (30) will be satisfied (for all possible magnitudes and distributions of inplane forces) if $w = \text{constant}$. thus if the interpolation functions for w include a constant term the complementary strain energy function, and hence matrix $[B]$ will no longer remain positive definite. On the other hand when only one of the forces is present, say $\bar{N}_{12}$, then any form of w which is made up of pure functions of x_1 and pure functions of x_2 will render the complementary strain energy and matrix $[B]$ non positive definite. Again if only $\bar{N}_{22}$ is present any form of w of the following form

$$w = G(x_1) + x_2 H(x_1)$$

where $G(x_1)$ and $H(x_1)$ are pure functions of x_1, will satisfy eqn (30) and will consequently render matrix $[B]$ singular.

Consider next the forms of w which will render the potential energy of inplane forces equal to zero. Such forms of w satisfy the equation

$$\int [\bar{N}_{11}(w_{,1})^2 + \bar{N}_{22}(w_{,2})^2 + 2\bar{N}_{12} w_{,1} w_{,2}] \, \mathrm{d}x_1 \, \mathrm{d}x_2 \equiv 0.$$

Now when all three inplane forces are present it can be seen that, w = constant, will yield zero potential energy and consequently matrix $[C]$ will become singular. It will be recalled that the same form of w will render matrix $[B]$ singular and therefore unless the constant term is excluded from the interpolation functions of w, the matrix $[E]$ will become singular for any value of λ. Likewise if only $\bar{N}_{22}$ is present the interpolation functions for w must be void of pure functions of x_1, otherwise both matrices $[B]$ and $[C]$ will become singular. In this case it will be recalled that the presence of functions of the form $w = x_2 H(x_1)$ will render matrix $[B]$ singular, but they will not effect the positive definiteness of matrix $[C]$ and consequently matrix $[E]$ will remain non singular. As a last example consider when only $\bar{N}_{12}$ is present. In this case the potential energy function is not definite in form and hence matrix $[C]$ will not be positive definite. As such it is necessary to require U^* and hence the matrix $[B]$ to become positive definite. This is achieved simply by dropping pure functions of x_1 and pure functions of x_2 from the interpolation functions of the interior displacement. It is apparent now that the interpolation functions for the interior displacement take slightly different forms according to the type of inplane loading. This calls for a simple logic in the programming of the element.

AN ILLUSTRATIVE EXAMPLE

We form a simple rectangular element by employing the following interpolation functions.

$$U_1 = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \beta_4 x_1^2 + \beta_5 x_1^2 x_2 + \beta_6 x_2^2 + \beta_7 x_1 x_2^2 + \beta_8 x_1^3 + \beta_9 x_2^3 \tag{33}$$

$$U_2 = -\beta_2 x_1 + \beta_{10} x_2 + \beta_{11} x_1 x_2 + \beta_{12} x_1^2 + \beta_{13} x_1^2 x_2 + \beta_{14} x_2^2 + \beta_{15} x_1 x_2^2 + \beta_{16} x_1^3 + \beta_{17} x_2^3 \tag{34}$$

$$w = \beta_{18} x_1 + \beta_{19} x_2 + \beta_{20} x_1 x_2 + \beta_{21} x_1^2 + \beta_{22} x_2^2. \tag{35}$$

In these interpolation functions we have dropped the constant terms of U_1, U_2 and w in accordance with comments made earlier. The resulting interpolation functions for the moments, as determined from eqns (13) are then full quadratic polynomials involving 21 β's. The complementary strain energy function, given by

$$U^* = \frac{1}{2D(1-\nu^2)} \int_{A_p} [M_{11} M_{22} M_{12}] \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & & 2(1+\nu) \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} \mathrm{d}A \tag{36}$$

may now be discretised and expressed in terms of β_M and β_w as

$$U^* = \frac{1}{2D(1-\nu^2)} [\beta_M \beta_w] \begin{bmatrix} B_{MM} & \lambda B_{Mw} \\ \lambda B_{Mw}^t & \lambda^2 B_{ww} \end{bmatrix} \begin{bmatrix} \beta_M \\ \beta_w \end{bmatrix}. \tag{37}$$

Likewise the potential energy of the inplane forces can be evaluated in terms of the last 5 β's as

$$V = \frac{1}{2} \int_{A_p} \bar{N}_{ij} w_{,i} w_{,j} \,\mathrm{d}A = \frac{1}{2} [\beta_M \beta_w] \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \lambda C_{ww} \end{array}\right] \begin{bmatrix} \beta_M \\ \beta_w \end{bmatrix}. \tag{38}$$

To determine matrix $[R]$, one must evaluate the work done along the element's boundary. This work term appears in eqn (7) in the form of an integral and a sum over s_p. The integral involves the normal moment and the *effective* shear force, and the sum involves the corner forces. It is, however, more convenient to express the boundary work in an alternative form wherein the corner forces do not appear explicitly and the work of the effective shear force is replaced by that of the shear force and the twisting moment. Thus we write,

$$\text{Boundary work} = \int_{sp} (Q_n w_b - M_{nt} w_{b,t} - M_{nn} w_{b,n}) \,\mathrm{d}s \tag{39}$$

where Q_n is the shear force along the edge. For a rectangular element, as shown in Fig. 2, we may write the discrete form of work done as

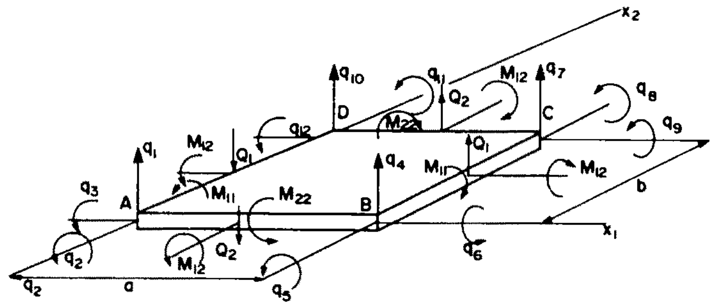

Fig. 2. Forces and nodal displacement of an element.

$$\text{Boundary work} = \{\beta\}^t[R]\{q\}$$

where

$$[R] = \oint_{sp} [J]^t[L]\,\mathrm{d}s.$$

The matrix $[J]$ contains the boundary force quantities and, for the rectangular element shown, it is given by

$$[J]^t = [J^t_{AB} J^t_{BC} J^t_{CD} J^t_{DA}]$$

where

$$J_{AB} = \begin{bmatrix} -Q_2 \\ M_{12} \\ M_{22} \end{bmatrix} \quad J_{BC} = \begin{bmatrix} Q_1 \\ -M_{11} \\ -M_{12} \end{bmatrix} \quad J_{CD} = \begin{bmatrix} Q_2 \\ -M_{12} \\ -M_{22} \end{bmatrix} \quad J_{DA} = \begin{bmatrix} -Q_1 \\ M_{11} \\ M_{12} \end{bmatrix}.$$

It is worth noting that since Q_i are given by

$$Q_i = (M_{ij} - \bar{N}_{ij}w)_{,j}$$

they may be expressed in terms of stress functions U_1 and U_2 alone.

To obtain matrix $[L]$, the boundary displacement w_b, and normal slope $w_{b,n}$, were interpolated independently by cubic and linear polynomials respectively. In view of appearance of the twisting moment in the boundary work it is also necessary to express the edge tangential slope in terms of q's. This is done consistently by differentiating the interpolation function for w_b. Hence for an edge of the element, say AB, we write

$$w_b = \left[1 - 3\left(\frac{x_1}{a}\right)^2 + z\left(\frac{x_1}{a}\right)^2\right]q_1 + [x_1 - 2x_1^2 + x_1^3]q_2$$

$$+ \left[3\left(\frac{x_1}{a}\right)^2 - 2\left(\frac{x_1}{a}\right)^3\right]q_4 - [x_1^2 - x_1^3)]q_5$$

$$w_{b,x1} = \frac{6}{a}\left[\left(\frac{x_1}{a}\right)^2 - \left(\frac{x_1}{a}\right)\right]q_1 + \left[1 - 4\left(\frac{x_1}{a}\right) + 3\left(\frac{x_1}{a}\right)^2\right]q_2$$

$$+ \frac{6}{a}\left[\left(\frac{x_1}{a}\right) - \left(\frac{x_1}{a}\right)^2\right]q_4 + \left[3\left(\frac{x_1}{a}\right)^2 - 2\left(\frac{x_1}{a}\right)\right]q_5$$

$$w_{b,x_2} = \left[1 - \left(\frac{x_1}{a}\right)\right]q + \left[\frac{x_1}{a}\right]q_6.$$

The element developed was employed for the buckling analysis of a square plate under different inplane force distributions and edge conditions. The results are shown in Figs 3–5. In

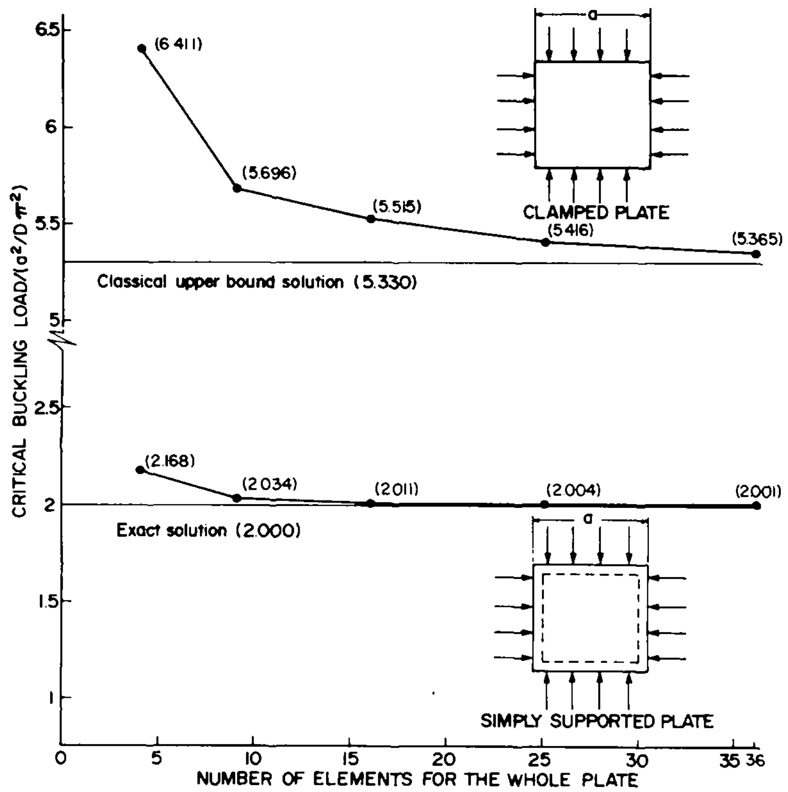

Fig. 3. Buckling of a square plate in biaxial compression.

all cases, the inplane force distributions were uniform. From these graphs it can be seen that the convergence to the correct solution is from above. However, this is not a property of the formulation and it depends upon the relative accuracy of the 3 sets of interpolation functions used for: the stress functions U_1, U_2; interior w; and boundary w_b and $w_{b,n}$. As such, the formulation provides considerable flexibility for developing a variety of elements. No attempt was made to arrive to an "optimum" set of interpolation functions.

In the case of uniaxial compression along the x_2 axis it was necessary to remove the pure functions of x_1 from the interior displacement function. In this case then, the interior displacement had 3 terms only. Attempts to analyse the case of pure shear loading failed. This is because in this case both pure x_1 and x_2 functions had to be removed leaving only the x_1x_2 term in the interior w. In this case it is readily seen that if the elemental axes are taken in the centre of the element, the potential energy of the shear force $\bar{N}_{12}$ (with the displacement field x_1x_2) will vanish identically. Thus, to analyse the case of pure shear, the interior displacement must contain higher degree mixed terms.

To ensure that the formulation was sound, the case of biaxial and shear loading was considered. In this case the interior displacement has 5 terms and as can be seen in Fig. 5 the element converges but its performance is not impressive.

Our primary aim here has been to demonstrate that hybrid elements can be developed for buckling analysis of plate and shell structures. We have also brought to light some matters of fundamental nature, in the development of such hybrid elements. the development of a very accurate element has not been of direct concern to us, and we have compared our computed results with those given in Timoshenko's text[13]. Nevertheless, to put the present element in perspective in Table 1, we compare some of our results with those obtained by other authors using alternative finite element formulations. All the results are for the 4×4 uniform mesh of a square plate. The elements developed by Kapur and Hartz[14], Dawe[15] and Carson and Newton[16], are all rectangular and are based on the displacement formulation but only the latter is a fully compatible element. The element developed by Tabarrok and Simpson[8] is also rectangular but it is based on the complementary energy principle. The element developed by Allman is triangular and it is based on a mixed variational principle. Allman's procedure has

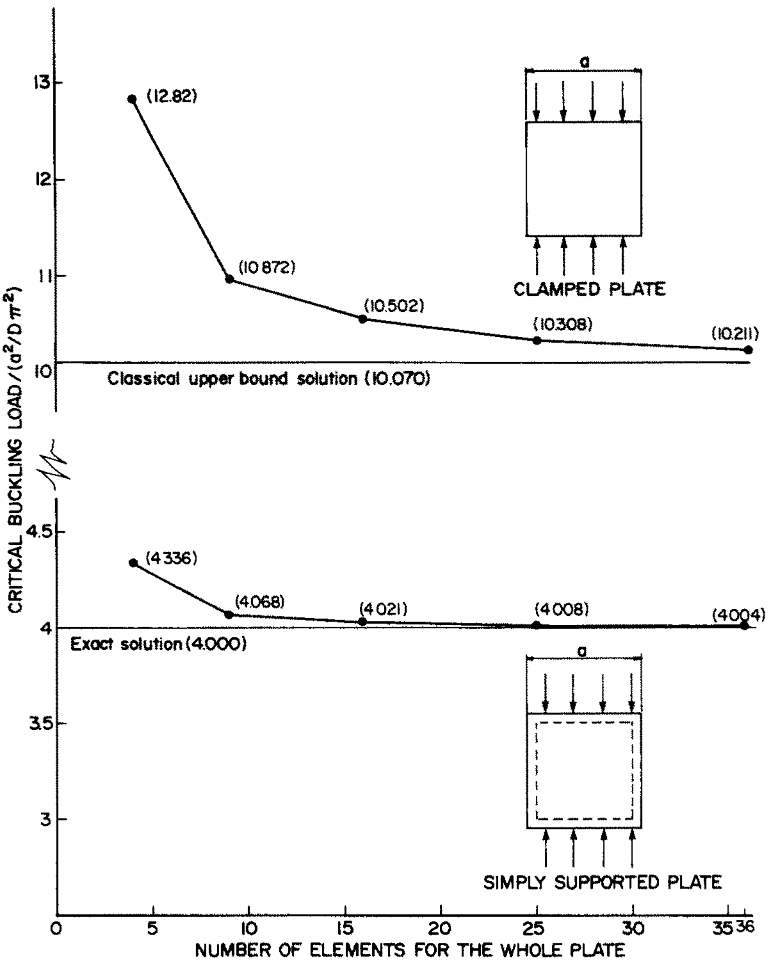

Fig. 4. Buckling of a square plate in uniaxial compression.

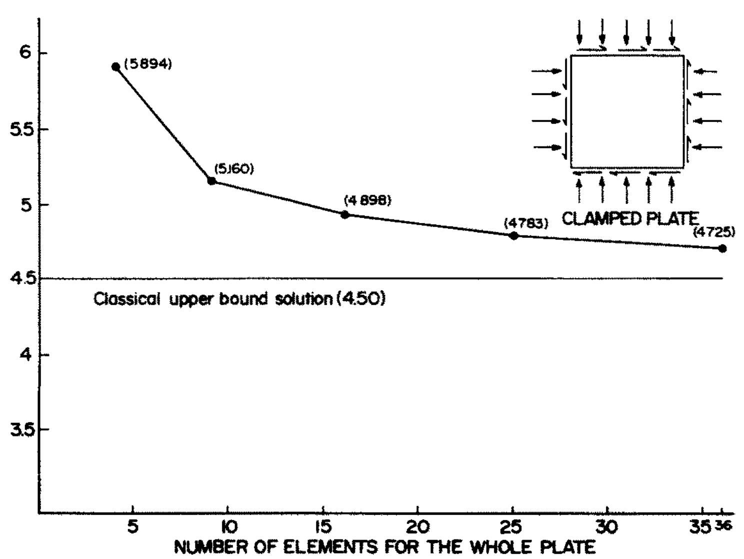

Fig. 5. Buckling of a square plate under uniform axial and shear loading.

Table 1. Critical loads of a square plate of side length l, under uniform plane stresses

Loading	Conditions	Ref. [13]	Ref. [8]	Ref. [14]	Ref. [15]	Ref. [16]	Ref. [5]	Present analysis
Uniform uniaxial compression	Simply supported	4.000	4.000	3.770	3.978	4.001	4.031	4.021
	Clamped	10.070†	10.123	9.782	10.065	—	10.990	10.502
Uniform biaxial compression	Simply supported	2.000	2.000	—	1.989	—	2.016	2.011
	Clamped	5.33	5.342	4.975	—	5.327	5.602	5.514

Multiplier = $\Pi^2 D/l^2$ for all cases.
†Upper bound solutions

some similarities to that presented here. Thus, as in the present formulation, Allman uses an independent set of interpolation functions for the moments, inside displacement, and boundary displacements and normal slopes. The interpolation functions used for the moments are linear and as a result they make no contribution to the equilibrium equation. However, inside the elements the displacement functions used by Allman are cubic and are not constrained to satisfy the equilibrium equations. Thus unlike the present formulation the inside equilibrium equations are violated in Allman's formulation and they tend to be satisfied approximately through the process of extremisation.

In conclusion, it is in order to pass comments on some apparent shortcomings of the present formulation. Consider first the dependency of the choice of interior displacements on the type of inplane loading. While this feature throws light on the fundamental structure of large deflection analysis, it is not a desirable component in a finite element package. Its elimination through is quite straightforward. It simply requires that the interpolation functions for the inside displacement be void of pure x_1 and pure x_2 functions. In the present formulation this requirement is very easily accommodated since the number of interior displacement functions need not match the number of nodal displacements and slopes. Indeed, with the hybrid element one is not constrained to use polynomials for variables inside the elements.

Next, consider the form of the system eqn (28) which arises as a non-linear eigenvalue problem requiring a determinantal search. In general, solution algorithms for linear eigenvalue equations are preferred and widely used. However, in recent years considerable progress has been made in efficient algorithms for locating roots of determinants within prescribed limits. In this connection reference should be made to the sign counting method of Wittrick and Williams [17].

Let us now examine eqn (23) in more detail. Denoting $D(1-\nu^2)$ by $\bar{D}$ we may evaluate β_M, from the first equation of (23), as

$$\beta_M = -B_{MM}^{-1}[\lambda B_{MM}\beta_w + \bar{D}R_M q]. \tag{40}$$

Substituting into the second equation yields

$$\left(\lambda C_{ww} - \frac{\lambda^2}{\bar{D}} B_{ww}\right)\beta_w - \frac{\lambda}{\bar{D}} B_{Mw}^t[-B_{MM}^{-1}(\lambda B_{Mw}\beta_w + \bar{D}R_M q] = \lambda R_w q. \tag{41}$$

Or rearranging

$$\lambda\left[C_{ww} - \frac{\lambda}{\bar{D}}(B_{ww} - B_{Mw}^t B_{MM}^{-1} B_{Mw})\right]\beta_w = \lambda[R_w - B_{Mw}^t B_{MM}^{-1} R_M]q. \tag{42}$$

In this equation λ may be cancelled on both sides of the equation. However, in doing so we tacitly agree to discount zero and infinite values of λ.

Equation (42) may then be written as

$$(C_{ww} - \bar{\lambda}H)\beta_w = Gq \tag{43}$$

where $\bar{\lambda} = \lambda/\bar{D}$ and matrices H and G are evident on comparing eqns (43) and (42). Finally, on inverting the coefficient matrix of β_w we may express this vector in terms of q. Several points deserve attention at this point. First we note that the coefficient matrix of β_w, which has to be inverted for each guessed value of $\bar{\lambda}$, is of the size of the number of interpolation functions used for interior displacements. For the element developed here this size does not exceed 5×5 and hence its inversion does not create problems. Indeed for such a small size one may invert the matrix analytically thus avoiding repetitive inversions. The large inversion of B_{MM}, appearing in eqn (40) is independent of $\bar{\lambda}$ and need only be carried out once.

Now on substituting the expression for β_M from eqn (40) into eqn (22), we may simplify our hybrid functional to the following form

$$\pi_H' = \sum_p \beta_w{}^t[(C_{ww} - \bar{\lambda}H)\beta_w - Gq] + F^tq. \tag{44}$$

The difference between π_H' and π_H in eqn (22) is that for the former the extremisation conditions with respect to β_M have been implicitly satisfied. If the extremisation conditions with respect to β_w are also satisfied in π_H', i.e. if β_w is eliminated from π_H' via eqn (43), we will recover eqn (25) which is entirely in terms of q's. In this process the eigenvalue $\bar{\lambda}$ will be absorbed in the determinantal eqn (28) as discussed earlier. Alternatively, we may retain the β_w's in π_H' and thereby preserve the linear eigenvalue form. To do so, we write eqn (44) in the following form

$$\pi_H' = \sum_p [\beta_w{}^t q^t] \left[\begin{array}{c|c} C_{ww} - \bar{\lambda}H & \frac{G}{2} \\ \hline \frac{G^t}{2} & 0 \end{array}\right] \begin{bmatrix} \beta_w \\ q \end{bmatrix} - [0 \quad F^t] \begin{bmatrix} \beta_w \\ q \end{bmatrix}. \tag{45}$$

Standard assembly routines may now be employed to obtain the global equations from eqn (45). The final system equations, when $F \equiv 0$, will emerge in the following constrained linear eigenvalue form.

$$\left[\begin{array}{ccc|c} C_{wwI} & & & \\ & C_{wwII} & & \Gamma \\ & & \cdot & \\ \hline & \Gamma^t & & 0 \end{array}\right] \begin{bmatrix} \beta_w^* \\ \\ \\ q^* \end{bmatrix} = \bar{\lambda} \left[\begin{array}{ccc|c} H_I & & & \\ & H_{II} & & 0 \\ & & \cdot & \\ \hline & 0 & & 0 \end{array}\right] \begin{bmatrix} \beta_w^* \\ \\ \\ q^* \end{bmatrix} \tag{46}$$

where Γ is obtained from elemental G matrices in the process of assembly.

It is apparent then that a linear eigenvalue form can be preserved by carrying forward the β_w's to the system equations. However, the price paid for this preservation is questionable. Firstly, we note that the size of the system eqn (46) is larger than that in eqn (28) by virtue of the presence of β_w^*'s. the connection for β_w's is clearly a unit matrix since the global vector β_w^* is made up of those of the elements stacked one above the other. Also the eigenvalue form in eqn (46) is not of standard type and the zeroes along its diagonals can create problems in some algorithms. On the other hand, it is perceivable that the block diagonal nature of eqn (46) can be used to advantage in some algorithms.

The method of satisfying the equilibrium equation in terms of stress functions and the transverse displacement, as expressed in eqn (13), applies equally well for nonlinear large deflection analysis of plates. An account of such an analysis is given in Ref. [18]. Alternatively, one may use stress and displacement variables directly in satisfying the equilibrium equation. A discussion of such an approach has been given by Boland and Pian[19].

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